Estimating Discrete Choice Dynamic Programming Models

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Introduction to Part II

- The second part focuses on econometric implementation of discrete choice dynamic programming models.
- A simple machine replacement model is estimated using the nested fixed point algorithm (Rust, 1987).
  - For more applications, see Aguirregabiria and Mira (2010) and Keane, Todd, and Wolpin (2010).
- For audience with different backgrounds, I will briefly review
  1. Discrete choice models and
  2. Numerical methods for (i) maximum likelihood estimation and (ii) dynamic programming.

Outline

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The Random Utility Model

- Consider a static problem in which the agent chooses among J alternatives, such as transportation mode and brand choice.
- Assume that the choice-specific utility can be expressed as

$$V_{ij} = u_{ij} + \varepsilon_{ij} \qquad i = 1, \ldots, N,\ j = 0 \ldots, J,$$

where u_{ij} is a deterministic component while ε_{ij} is a stochastic component.

  - A simple example is $u_{ij} = x_{ij}\theta$, where x_{ij} is a vector of observed characteristics such as price and income.

- Let $A = \{1, \ldots, J\}$ denote the choice set. The choice probability is

$$P(a) = \int_{\varepsilon} 1\left\{ a = \arg\max_{j \in A} \left(u_{ij} + \varepsilon_{ij} \right) \right\} f(\varepsilon_i)\, d\varepsilon_i,$$

where $f(\varepsilon_i)$ is a joint density of $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iJ})'$.

The Conditional Logit Model

- Assume that each ε_j is independently, identically distributed extreme value. The density is $f(\varepsilon_j) = e^{-\varepsilon_j} \exp(-e^{-\varepsilon_j})$. The choice probability follows the logit model (McFadden, 1981).

$$p_{ij} = P(a = j) = \frac{\exp(u_{ij})}{\sum_{j=1}^{J} \exp(u_{ij})} = \frac{\exp(u_{ij} - u_{i1})}{1 + \sum_{j=2}^{J} \exp(u_{ij} - u_{i1})}$$

- Note that this is a system of $J - 1$ equations. For $J = 2$,

$$p_{i2} = \frac{\exp(u_{i2} - u_{i1})}{1 + \exp(u_{i2} - u_{i1})} = \Lambda(u_{i2} - u_{i1}).$$

There exists a mapping between the utility differences and choice probabilities.

$$u_{i2} - u_{i1} = \Lambda^{-1}(p_{i2}) \qquad \text{or} \qquad u_{i2} - u_{i1} = \ln\left(\frac{p_{i2}}{1 - p_{i2}}\right)$$

See Hotz and Miller (1993) for the details on invertibility.

Outline

MLE

- The choice probability is

$$P(a=j)=\frac{\exp(u_{ij})}{\sum_{j=1}^{J}\exp(u_{ij})}$$

- Suppose $V_{ij}=x_{ij}\theta+\varepsilon_{ij}$. The log-likelihood function is

$$l(\theta)=\sum_{i=1}^{N}l_i(\theta)=\sum_{i=1}^{N}\sum_{j=1}^{J}1(a_i=j)\ln P(a_i=j|x_{ij}).$$

- If the model is correctly specified,

$$\widehat{\theta}\xrightarrow{d}\mathcal{N}\left(\theta,\mathbb{E}\left(-\frac{\partial^2 l}{\partial\theta\partial\theta'}\right)^{-1}\right).$$

Numerical Maximization (Train, 2009)

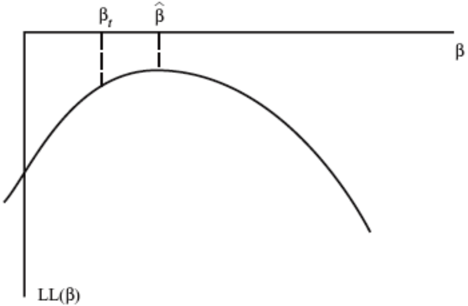

Figure 8.1. Maximum likelihood estimate.

Numerical Maximization (Train, 2009)

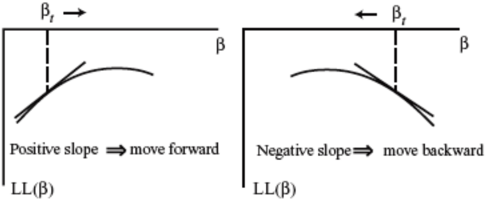

Figure 8.2. Direction of step follows the slope.

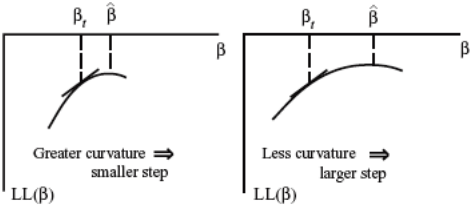

Figure 8.3. Step size is inversely related to curvature.

Numerical Methods for MLE

- The Newton-Raphson procedure uses the following formula.

$$\theta_{k+1} = \theta_k + \left(-H_k^{-1}\right) g_k,$$

where $g_k = \left. \partial l / \partial \theta \right|_{\theta_k}$ and $H_k = \left. \partial^2 l / \partial \theta \partial \theta' \right|_{\theta_k}$.

- Quasi-Newton methods:
  $[\texttt{fval}, \texttt{theta}, \texttt{g}, \texttt{H}] = \texttt{csminwel}(\texttt{'fun'}, \texttt{theta0}, \texttt{H0}, \texttt{g}, \texttt{crit}, \texttt{nit})$ by Christopher Sims

- Non-gradient based algorithm (simplex methods):
  $[\texttt{theta}, \texttt{fval}] = \texttt{fminsearch}(\texttt{'fun'}, \texttt{theta0}, \texttt{options})$

Extension to Dynamic Models

- We may be concerned with dynamics in some applications such as firm's adjustment of labor and capital, educational and occupational choices.
- The value function V is the unique solution to the Bellman equation:

$$V(x_t, \varepsilon_t) = \max_{j \in A} \{ V_{jt} = V(x_t, \varepsilon_t, a_t = j) \},$$

where the choice-specific value function (including ε) is

$$\begin{aligned} V(x_t, \varepsilon_t, a_t) &= u(x_t, a_t) + \varepsilon_t(a_t) + \beta \mathbb{E}[V(x_{t+1}, \varepsilon_{t+1}) | x_t, \varepsilon_t, a_t] \\ &= \{u(x_t, a_t) + \beta \mathbb{E}[V(x_{t+1}, \varepsilon_{t+1}) | x_t, \varepsilon_t, a_t]\} + \varepsilon_t(a_t) \\ &= v(x_t, a_t) + \varepsilon_t(a_t) \end{aligned}$$

for $\beta \in (0, 1)$. The model considered here is a natural extension of the conditional logit model (or the random utility model) to a dynamic setting.

Outline

Plant Investment Decisions

- Here we consider the plant's investment decision problem set out by Kasahara at UBC. The tradeoff is that investment (machine replacement), $a \in \{0,1\}$,
  1. requires large fixed costs for now but
  2. increases profits and lowers replacement costs in the future.
- The profit function is

$$u(z_t, a_{t-1}, a_t) = R(z_t, a_t) - FC(a_t, a_{t-1}),$$

  where R is revenues net of variable input costs, z_t is plant's productivity, and FC is fixed costs.
- Assume that $z_t = \rho z_{t-1} + \eta_t$, where $\eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$. Specify the profit function as

$$\begin{aligned} u(z_t, a_{t-1}, a_t) &= u(z_{t-1}, a_{t-1}, a_t) \\ &= \exp(\theta_0 + \theta_1 z_{t-1} + \theta_2 a_t) - [\theta_3 a_t + \theta_4 (1 - a_{t-1}) a_t]. \end{aligned}$$

Transition Probability

- Let $Z = \{z_1, \ldots, z_g, \ldots, z_G\}$.
- Denote the transition probability by

$$q_{gh} = q\left(z_t = z_h \middle| z_{t-1} = z_g\right) = \Pr\left(z \in [z_h, z_{h+1}] \middle| z \in [z_g, z_{g+1}]\right).$$

By the Tauchen method, this can be approximated by

$$q_{gh} = \begin{cases} \Phi\left(\left(\bar{z}_1 - \rho z_g\right)/\sigma_\eta\right) & \text{if } h = 1, \\ \Phi\left(\left(\bar{z}_h - \rho x_g\right)/\sigma_\eta\right) - \Phi\left(\left(\bar{z}_{h-1} - \rho z_g\right)/\sigma_\eta\right) & \text{if } 1 < h < G, \\ 1 - \Phi\left(\left(\bar{z}_G - \rho z_g\right)/\sigma_\eta\right) & \text{if } h = G, \end{cases}$$

where $\bar{z}_g = (z_g + z_{g+1})/2$ and Φ is the standard normal cdf.

The Choice-Specific Value Function

- Define the integrated value function as

$$\overline{V}(z_{t-1}, a_{t-1}) = \int_{\varepsilon_t} V(z_{t-1}, a_{t-1}, \varepsilon_t)\, dF(\varepsilon_t).$$

- Assume that each $\varepsilon_t(a_t)$ is iid extreme value.

$$\overline{V}(z_{t-1}, a_{t-1}) = \gamma + \ln\left(\sum_{j=1}^{J} \exp(v(z_{t-1}, a_{t-1}, a_t = j))\right),$$

  where γ is the Euler constant.

- The choice-specific value function is

$$v(z_{t-1}, a_{t-1}, a_t) = u(z_{t-1}, a_{t-1}, a_t) + \beta \sum_{x_t} q(z_t | z_{t-1}) \overline{V}(z_{t-1}, a_{t-1}).$$

Outline

Infinite Horizon Problem

- In the infinite-horizon problem with a finite number of states, the Bellman equation is a system of nonlinear equations.

$$V_g = \max_{a \in A} \left[u(x_g, a) + \beta \sum_{h=1}^{G} q_{gh}(a) V_h \right],$$

where $a \in \{1, \ldots, J\}$ is a choice variable, $x \in \{x_1, \ldots, x_g, , \ldots, x_G\}$ is a state variable, and $q_{gh}(a)$ is the transition probability from state g to state h.

- Let $\mathbf{a} = (a_1, \ldots, a_G)'$ denote the policy function and $\mathbf{u}^{\mathbf{a}} = (u(x_1, a_1), \ldots, u(x_G, a_G))'$ the return. Then, in vector notation, $\mathbf{V}^{\mathbf{a}} = \mathbf{u}^{\mathbf{a}} + \beta \mathbf{Q}^{\mathbf{a}} \mathbf{V}^{\mathbf{a}}$, which leads to the solution:

$$\mathbf{V}^{\mathbf{a}} = (\mathbf{I} - \beta \mathbf{Q}^{\mathbf{a}})^{-1} \mathbf{u}^{\mathbf{a}},$$

where the ghth element of $\mathbf{Q}^{\mathbf{a}}$ is $q_{gh}(a)$. For details, see Judd (1998) and Adda and Cooper (2003).

Value Iteration Algorithm

1. Choose a grid $X = \{x_1, \ldots, x_g, \ldots, x_G\}$, where $x_g < x_h$ for $g < h$, and specify $u(x, a)$ and $q_{gh}(a)$.
2. Make an initial guess V^0 and choose stopping criterion $c > 0$.
3. For $g = 1, \ldots, G$, compute

$$V_g^{k+1} = \max_{a \in A} \left\{ u(x_g, a) + \beta \sum_{h=1}^{G} q_{gh}(a) V_h^k \right\}.$$

4. If $\left\| V^k - V^{k+1} \right\| < c$, stop; else go to step 3.

Policy Iteration Algorithm

1. Choose a grid $X = \{x_1, \ldots, x_g, \ldots, x_G\}$, where $x_g < x_h$ for $g < h$, and specify $u(x, a)$ and $q_{gh}(a)$.
2. Make an initial guess V^0 and choose stopping criterion $c > 0$.
3. Compute $\mathbf{a}^{k+1} = \left(a_1^{k+1}, \ldots, a_G^{k+1}\right)'$, where

$$a_g^{k+1} = \arg\max_{a \in A} \left\{ u(x_g, a) + \beta \sum_{h=1}^{G} q_{gh}(a) V_h^k \right\}.$$

4. Compute $\mathbf{u}^{k+1} = \left(u\left(x_1, a_1^{k+1}\right), \ldots, u\left(x_G, a_G^{k+1}\right) \right)'$.
5. Compute

$$\mathbf{V}^{k+1} = \left(\mathbf{I} - \beta \mathbf{Q}^{\mathbf{a}^{k+1}}\right)^{-1} \mathbf{u}^{k+1}.$$

6. If $\left\| V^k - V^{k+1} \right\| < c$, stop; else go to step 3.

Outline

The Nested Fixed Point Algorithm (Rust, 1987)

- The conditional choice probability takes the form:

$$P(a|x) = \frac{\exp\left(u(x,a,\theta) + \beta \mathbb{E} V(x',\varepsilon')\right)}{\sum_{j=1}^{J} \exp\left(u(x,j,\theta) + \beta \mathbb{E} V(x',\varepsilon')\right)}.$$

- The transition probability can be obtained separately.

1. Inner loop: Given θ, compute the value function V using the value iteration algorithm or the policy iteration algorithm, and obtain the conditional choice probability P.
2. Outer loop: Given P, obtain a pseudo-likelihood estimate of θ using the Newton method.
3. Repeat steps 1 and 2 until convergence.

Outline

Hotz and Miller (1993) and Aguirregabiria and Mira (2002)

- Denote the optimal decision rule by $d_{jt}^{o} = 1\{a_t = j | x_t, \varepsilon_t\}$.

$$V(x_t) = \mathbb{E}\left\{\sum_{\tau=t}^{T}\sum_{j=1}^{J}\beta^{\tau-t} d_{jt}^{o}\left[u(x_t, a_t) + \varepsilon_t(a_t)\right]\right\}$$

The Bellman equation can be rewritten as

$$V(x) = \sum_{a} P(a|x)\left[u(x,a) + \mathbb{E}[\varepsilon | x, a] + \beta \sum_{x'} q(x'|x,a) V(x')\right].$$

- The system of equations can be solved for the value function. In matrix notation,

$$V = \left(I - \beta \sum_{a} P(a) \circ Q(a)\right)^{-1} \sum_{a} P(a) \circ \left[u(a) + (\gamma - \ln P(a))\right],$$

where $\circ$ is the element-by-element product.

The Conditional Logit Form

- Suppose $u(x,a) = z(x,a)\alpha$. The policy iteration operator is

$$\Psi(P) = \frac{\exp(\widetilde{z}(x,a)\alpha + \widetilde{e}(x,a))}{\sum_{j=1}^{J}\exp(\widetilde{z}(x,a)\alpha + \widetilde{e}(x,a))},$$

where

$$\begin{aligned}
\widetilde{z}(x,a) &= z(x,a) + \beta\sum_{x'} q(x'|x,a)\, W_z(x',P), \\
\widetilde{e}(x,a) &= \beta\sum_{x'} q(x'|x,a)\, W_e(x',P), \\
W_z(x',P) &= \left[I - \beta\sum_{j} P \circ Q\right]^{-1} \sum_{a} P \circ z, \\
W_e(x',P) &= \left[I - \beta\sum_{a} P \circ Q\right]^{-1} \sum_{a} P \circ (\gamma - \ln(P)).
\end{aligned}$$

The Nested Pseudo Likelihood Algorithm

- Make an initial guess P^0 and choose the number of iterations K.

1. Given $\widetilde{z}(x,a) = z(x,a) + \beta \sum_{x'} q(x'|x,a) W_z(x', P^{k-1})$ and $\widetilde{e}(x,a) = \beta \sum_{x'} q(x'|x,a) W_e(x', P^{k-1})$, obtain a pseudo-likelihood estimate of α^k as

$$a^k = \arg\max_{\alpha} \sum_{i=1}^{N} \Psi\left(P^{k-1}\right).$$

2. Given a^k, update P using

$$P^k = \Psi\left(P^{k-1}\right).$$

3. Iterate steps 1 and 2 until $k = K$.

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